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ON 3-MANIFOLDS THAT HAVE FINITE FUNDAMENTAL GROUP AND CONTAIN KLEIN BOTTLES

BY

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ABSTRACT. The closed irreducible 3-manifolds with finite fundamental group and containing an embedded Klein bottle can be identified with certain Seifert fibre spaces. We calculate the isotopy classes of homeomorphisms of such 3-manifolds. Also we prove that a free involution acting on a manifold of this type, gives as quotient either a lens space or a manifold in this class. As a corollary it follows that a free action of Z_8 or a generalized quaternionic group on S^3 is equivalent to an orthogonal action.

0. Introduction. We are in the PL category. The object of study is the class of closed, irreducible orientable 3-manifolds which contain embedded Klein bottles and have finite fundamental group. These 3-manifolds are easily shown to be exactly the Seifert fibre spaces [7] with at most 3 exceptional fibres of multiplicity $2, 2, p$ $(p \geqslant 1)$ and the 2-sphere as orbit surface, excluding $S^2 \times S^1$.

We prove that any homeomorphism homotopic to the identity is isotopic to the identity for such a 3-manifold M (this was done for a particular case where $p = 2$ in [4]). Also the factor group of the group of orientation-preserving homeomorphisms of M by the normal subgroup of homeomorphisms isotopic to the identity, which is denoted $\mathcal{H}(M)$, is shown to be one of the groups Z_2, $Z_2 + Z_2$, S_3 and $S_3 + Z_2$. There are no orientation-reversing homeomorphisms of M.

Finally we establish that any free involution on M gives as quotient either a lens space or a 3-manifold in the above class. Let $Q(8m)$ be the group $\{x, y | x^2 = (xy)^2 = y^{2m}\}$. As a corollary it follows that a free action of $Q(2^k)$ on S^3, $k \geqslant 3$, is equivalent to an orthogonal action. Also simpler proofs of the analogous result in [5] and [6] for Z_4 and Z_8 are given.

Note that the 3-manifolds in the above class are not sufficiently large. Therefore it is interesting to see that some of the results of Waldhausen [9] can be achieved in this case. In another paper [11] we will build on the work here to obtain that free actions of some finite groups of order $2^m 3^n$ on S^3 are equivalent to orthogonal actions.

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1. Seifert spaces.

DEFINITION. A closed surface J embedded in a 3-manifold M is incompressible if (1) J is a 2-sphere and J does not bound a 3-cell or (2) J is not a 2-sphere and there is no disk D embedded in M with $D \cap J = \partial D$ a noncontractible curve in J.

LEMMA 1. *Let K be a Klein bottle. Then there are exactly five isotopy classes of simple closed curves in K. If $\pi_1(K) = \{a, b | b^{-1}ab = a^{-1}\}$ then these are represented by $\{1\}$, a, b, ab, b^2.*

PROOF. See **[4]**.

Let M be a closed, irreducible orientable 3-manifold with finite fundamental group and K be an embedded Klein bottle in M. Since M is orientable, K must be one-sided in M. We denote a small regular neighbourhood of K by N. Finally let $Y = M - \operatorname{int} N$ and denote $\partial Y = \partial N$ by L.

LEMMA 2. *K is incompressible and Y is a solid torus.*

PROOF. Suppose that K is compressible in M and let D be a disk with $D \cap K = \partial D = C$ noncontractible in K. Then C is two-sided in K and therefore either is a nonseparating curve on K or divides K into two Möbius bands (cf. Lemma 1). Let $N(D)$ be a small regular neighbourhood of D, which intersects K in an annulus A. Let D_0 and D_1 be the two disjoint disks in $\partial N(D)$ with $\partial D_0 \cup \partial D_1 = \partial A$. If we replace K by $(K - \operatorname{int} A) \cup D_0 \cup D_1$ then the result is either a nonseparating 2-sphere (since K is one-sided) or two disjoint one-sided projective planes in M. Both of these possibilities contradict $\pi_1(M)$ is finite. So K must be incompressible.

Since $\pi_1(M)$ is finite, by Lemma 14.12 of **[12]** it follows that Y is a handlebody as desired (i.e. a solid torus).

PROPOSITION 3. *The class of Seifert spaces with S^2 as orbit surface and at most 3 exceptional fibres of multiplicity 2, 2, p ($p > 1$) excluding $S^2 \times S^1$, is equivalent to the class of irreducible 3-manifolds which have finite fundamental group and contain an embedded Klein bottle.*

PROOF. Suppose M is of the latter type. $\pi_1(L)$ has generators given by a and b^2 in $\pi_1(K)$. N can be fibered by circles which have homotopy class b^2, with two exceptional fibres of multiplicity 2 at the centres of the Möbius bands on K (with classes b and ab). Since K is incompressible, the boundary of a meridian disk for Y yields an element of $\pi_1(L)$ different from b^2. So the fibering extends to Y with another exceptional fibre of multiplicity p ($p > 1$).

Conversely let M be a Seifert fibre space as in the proposition. If λ is a nonsingular arc in the orbit surface, joining the images of the exceptional fibres of multiplicity 2 and missing the image of the other exceptional fibre, then the set of points of M which project to λ form a Klein bottle. Since M is not homeomorphic to $S^1 \times S^2$ it follows that $\pi_1(M)$ is finite and M has S^3 as its universal cover. Therefore M is irreducible and the result is proved.

Suppose M is a 3-manifold satisfying the conditions in Proposition 3. Let D be a meridian disk for Y and let $C = \partial D$. Assume the homotopy class $\{C\} = a^m b^{2n}$, where $m, n > 0$ and $(m, n) = 1$. Then $\pi_1(M)$ has the presentation $\{a, b | b^{-1}ab = a^{-1}, a^m b^{2n} = 1\}$. Since K is incompressible, $m \neq 0$ and $n \neq 0$. Conjugating $a^m b^{2n} = 1$ by b, we see that $a^{2m} = b^{4n} = 1$. Let $4n = 2^k n_1$ where n_1 is odd, and let b_1 denote b^{n_1}. Then $\pi_1(M) = Z_{n_1} \times G$ where the cyclic group has generator b^{2^k} and $G = \{a, b_1 | b_1^{-1} a b_1 = a^{-1}, a^m b_1^{2^{k-1}} = 1\}$.

If m is odd then $G = D(2^k, m) = \{a_1, b_1 | b_1^{-1} a_1 b_1 = a_1^{-1}, a_1^m = 1, b_1^{2^k} = 1\}$, where $a_1 = a^2$. If m is even then since $(m, n) = 1$ it follows that n is odd, $k = 2$ and $n_1 = n$. In this case $G = Q(4m) = \{a, b_1 | b_1^2 = (ab_1)^2 = a^m\}$.

In the degenerate case $m = 1$, clearly $\pi_1(M) = Z_{4n}$. By **[1]**, $M = L(4n, \pm(2n - 1))$ since M contains a Klein bottle.

2. The homeotopy group. Let M be a 3-manifold with the properties in Proposition 3, throughout this section.

THEOREM 4. *If h: $M \to M$ is any homeomorphism with $h_\#: H_1(M, Z_2) \to H_1(M, Z_2)$ equal to the identity, then h is isotopic to a map taking K to K.*

PROOF. Denote $h(K)$ by K' and assume that K' and K are transverse. Since $h_\# = \text{id}$, $h_*: \pi_1(M) \to \pi_1(M)$ must preserve the normal subgroup G of index 2 obtained from the orientation-preserving elements of $\pi_1(K)$. (Note that commutators in $\pi_1(K)$ are orientation-preserving loops.) Since the image of $\pi_1(Y)$ in $\pi_1(M)$ is clearly G, it follows that $K' \cap Y$ must be orientable.

By the incompressibility of K' and K, and the irreducibility of M, there is an obvious isotopy of K' eliminating all the contractible curves of intersection of K' and K. Consequently it suffices to suppose that $K' \cap Y$ contains annuli only and all the curves of $K' \cap L$ are noncontractible and parallel on L. By the well-known fact that a properly embedded, incompressible annulus in a solid torus is parallel into the boundary, we can then find an isotopy of K' achieving $K' \cap Y = \varnothing$.

Let N' be a small regular neighbourhood of K' in N and let $L' = \partial N'$. If the map $\pi_1(L') \to \pi_1(N)$ has nontrivial kernel then the argument in Lemma 14.12 of **[12]** implies that M is contained in N, which is impossible. So L' is incompressible in N, and letting $W = N - \text{int}\, N'$ we see that W is an h cobordism. Therefore W is homeomorphic to $S^1 \times S^1 \times I$ (cf. **[8]**) and there is an isotopy taking L' to L. Using [9] we can achieve $K' = K$ by another isotopy, since N is sufficiently large.

THEOREM 5. *If $h: M \to M$ is a homeomorphism homotopic to the identity then h is isotopic to the identity.*

PROOF. By Theorem 4 it suffices to assume h takes K to itself. Suppose h fixes the base point on K. Then $h_*: \pi_1(K) \to \pi_1(K)$ maps a to $a^{\pm 1}$ and b to $b^{\pm 1}$ or $(ab)^{\pm 1}$ without loss of generality, by Lemma 1. There is an isotopy in K inducing conjugation of $\pi_1(K)$ by b. This takes a to a^{-1} and so we can assume $h_*(a) = a$.

As h is homotopic to the identity, b and $h_*(b)$ are conjugate in $\pi_1(M)$. Therefore for some element g, $b^{-1}g^{-1}h_*(b)g$ is in the normal closure of the relation $r = a^m b^{2n}$ in $\pi_1(K)$. By a calculation in $\pi_1(K)$, one sees that $g^{-1}h_*(b)g = h_*(b)a^{2i}$ for some integer i. So

$$b^{-1}h_*(b)a^{2i} = g_1^{-1}r^{\pm 1}g_1 g_2^{-1}r^{\pm 1}g_2 \dots . \qquad (+)$$

Suppose $h_*(b) = b^{-1}$ or $(ab)^{-1}$. If we put $a = 1$ in $(+)$ then it follows that $n = 1$. On the other hand if we assume $h_*(b) = ab$ and set $a^2 = 1$ in $(+)$ then this gives a contradiction. Finally in the case that $h_*(b) = b$, $h: K \to K$ is homotopic to the identity. Therefore by [2], after an isotopy we obtain that h is the identity on K. Because h must be orientation-preserving it is easy to isotop h to the identity on N and then on all of M.

Assume now that $h_*(b) = b^{-1}$ or $(ab)^{-1}$ and $n = 1$, i.e., $\{\partial D\} = b^2a^m$ where D is a meridian disk for Y. Then the classes a and $\{\partial D\}$ have intersection number ± 1 in L. We isotop K as follows:

First we can move K till $K \cap Y$ is an annulus A in L, with the curves of ∂A having homotopy class a. The meridian disk D can be assumed to meet A at a single arc. Therefore A is parallel to $L - \operatorname{int} A$ in Y and there is an isotopy of K taking A to $L - \operatorname{int} A$. Then K can be shifted back to its original position, by the same argument as at the end of Theorem 4.

Depending on the direction of the isotopy, we see that b is transformed to the class $b(b^2a^m)^{\pm 1}$ in $\pi_1(K)$. For the appropriate choice, the result is $b^{-1}a^{-m}$. Consequently if the isotopy is applied to h then a homeomorphism is obtained which takes b to ba^p for some p. By the previous argument, this is isotopic to the identity as required.

THEOREM 6. *Let M be a 3-manifold as in Proposition 3. Then*

$$\mathcal{H}(M) = \begin{cases} Z_2 + Z_2 & \text{if } m \neq 2 \text{ and } n \neq 1, \\ Z_2 & \text{if } m \neq 2 \text{ and } n = 1, \\ S_3 + Z_2 & \text{if } m = 2 \text{ and } n \neq 1, \\ S_3 & \text{if } m = 2 \text{ and } n = 1. \end{cases}$$

There are no orientation-reversing homeomorphisms of M.

PROOF. Let the map $\mathcal{H}(M) \to \operatorname{Aut} H_1(M, Z_2)$ given by $h \to h_\#$ have kernel $\mathcal{G}$. By Theorem 4, a homeomorphism h with isotopy class in $\mathcal{G}$ can be assumed to map K to itself. By Lemma 1, without loss of generality h_*: $\pi_1(K) \to \pi_1(K)$ takes a to $a^{\pm 1}$ and b to $b^{\pm 1}$ or $(ab)^{\pm 1}$. Conversely the homeomorphisms of K which transform the pair (a, b) to one of (a, b), (a^{-1}, b^{-1}), (a, ab), $(a^{-1}, (ab)^{-1})$ clearly map $\{\partial D\}$ to $\{\partial D\}^{\pm 1}$ and so extend to homeomorphisms of M. Since there is an isotopy of K taking a to a^{-1} these maps give all possible isotopy classes in $\mathcal{G}$.

Suppose first that m is odd. Then $H_1(M, Z_2) = Z_2$ and so $\mathcal{G} = \mathcal{H}(M)$. The argument in Theorem 5 shows that no pair of the elements $b^{\pm 1}$, $(ab)^{\pm 1}$ are conjugate in $\pi_1(M)$ for $n \neq 1$, and so $\mathcal{H}(M) = Z_2 + Z_2$. On the other hand if $n = 1$ then a homeomorphism h with $h(K) = K$ and $h_*(b) = (ab)^{-1}$ is isotopic to the identity (by the method in Theorem 5). Therefore $\mathcal{H}(M) = Z_2$ in this case.

Assume now that m is even. Then $H_1(M, Z_2) = Z_2 + Z_2$ and a homeomorphism h taking K to K with $h_*(b) = (ab)^{\pm 1}$ induces a nontrivial involution in $\operatorname{Aut} H_1(M, Z_2)$. Therefore the same process as in the previous paragraph shows that $\mathcal{G} = Z_2$ if $n \neq 1$ and $\mathcal{G} = \{1\}$ if $n = 1$.

Let $\mathcal{G}_0$ be the quotient of $\mathcal{H}(M)$ by $\mathcal{G}$. $\mathcal{G}_0$ is isomorphic to the image of $\mathcal{H}(M)$ in $\operatorname{Aut} H_1(M, Z_2)$ and we already know the latter group contains an element of order 2. So $\mathcal{G}_0 = Z_2$ or S_3 are the only possibilities. If the latter holds then there is a homeomorphism h: $M \to M$ with $h_\# \in \operatorname{Aut} H_1(M, Z_2)$ of order 3. Assume h_*: $\pi_1(M) \to \pi_1(M)$ takes a to a^ib^j. Then a^ib^j must have order $2m$. Consequently b^{2mj} is a power of a and so n divides j (since $(m, n) = 1$ and m is even). If j is odd then a^ib^j has order 4 and $m = 2$. If j is even then a^ib^j is a power of a and $h_\#$ is not of order 3. This establishes that for $m \neq 2$, $\mathcal{G}_0 = Z_2$.

Finally suppose $m = 2$. Then $\{\partial D\} = a^2b^{2n}$ and b^2 has intersection number ± 2 with $\{\partial D\}$ in L. Consequently there is a Möbius band B embedded properly in Y with ∂B having the homotopy class b^2. But it is clear that another Möbius band B_1 can be chosen in N with $\partial B_1 = \partial B$. So $B \cup B_1$ gives a Klein bottle K' in M.

By Lemma 2, $M = N' \cup Y'$ where N' is a small regular neighbourhood of K' and $Y' = M - \operatorname{int} N'$ is a solid torus. Let D' be a meridian disk for Y'. Then $\{\partial D'\} = a_0^m b_0^{2n}$ where $\pi_1(K') = \{a_0, b_0 | b_0^{-1} a_0 b_0 = a_0^{-1}\}$, since the numbers m, n are in 1-1 correspondence with the isomorphism class of the group $\pi_1(M)$. Therefore it is clear that a homeomorphism from K to K' can be found which extends to M, and so $\mathcal{G}_0 = S_3$.

For $m \neq 2$, $n = 1$ we obtain $\mathcal{H}(M) = \mathcal{G}_0 = Z_2$. If $m = 2$, $n = 1$ it follows that $\mathcal{H}(M) = \mathcal{G}_0 = S_3$. Finally suppose $n \neq 1$. Then $\mathcal{H}(M)$ contains a subgroup $Z_2 + Z_2$. Therefore if $m \neq 2$, $\mathcal{H}(M) = Z_2 + Z_2$ and if $m = 2$ then $\mathcal{H}(M) = S_3 + Z_2$ since this is the only nonabelian group which has order 12 and contains a normal subgroup Z_2 (with quotient S_3).

Suppose h: $M \to M$ is an orientation-reversing homeomorphism. If $h_\# \in \operatorname{Aut} H_1(M, Z_2)$ is of order 3 then we replace h by h^3. So it suffices to assume (by Theorem 4) that there is a Klein bottle K in M, so that after an isotopy of h, $h(K) = K$. Then if we compose h with a suitable orientation-preserving homeomorphism, a new h is obtained with $h = \mathrm{id}$ on K.

By the argument in the last paragraph of the proof of Theorem 4, we can adjust h so that also h: $N \to N$. Then since h is orientation-reversing, it must be the case that h: $L \to L$ is orientation-reversing. Suppose h_*: $\pi_1(L) \to \pi_1(L)$ maps a to a^ib^j and b^2 to a^qb^{2r}. Since $h = \mathrm{id}$ on K, it follows that in $\pi_1(K)$ the classes a and a^ib^j must be conjugate, and similarly for b^2 and a^qb^{2r}. By a calculation in $\pi_1(K)$, one sees that $i = \pm 1, j = 0, q = 0$ and $r = 1$. Then since h: $L \to L$ is orientation-reversing, we find that $i = -1$. But h_*: $\pi_1(L) \to \pi_1(L)$ maps $\{\partial D\}$ to $\{\partial D\}^{\pm 1}$, and $\{\partial D\} = a^mb^{2n}$ for $m > 0, n > 0$. This gives a contradiction.

3. 2-groups acting freely on S^3. In [3] it is proved that a free action of Z_2 on S^3 is equivalent to an orthogonal action. We begin with a simple demonstration of:

PROPOSITION 7 [5]. *Any free action of Z_4 on S^3 is equivalent to an orthogonal action.*

PROOF. By [3], the quotient of S^3 by the action of the subgroup Z_2 of Z_4 is RP^3. Let P be an embedded projective plane in RP^3. The action of Z_4 gives a free involution g on RP^3.

Assume without loss of generality that P and gP are transverse (cf. the lemma in [5]). $P \cap gP$ contains a loop which is one-sided in P and gP, and all the other components of $P \cap gP$ bound disks in both surfaces. This follows by Poincaré duality, since a one-sided curve in P gives an element of $H_1(RP^3, Z_2)$ dual to the class in $H_2(RP^3, Z_2)$ corresponding to gP.

Suppose C is a curve of $P \cap gP$ chosen so that C bounds a disk D in gP with $(\operatorname{int} D) \cap P = \varnothing$. Let $C = \partial D_1$ with D_1 in P. If C is g-invariant then $D_1 = gD$. Hence $D \cup D_1$ is a g-invariant sphere which bounds a g-invariant 3-cell in RP^3. By the Brouwer Fixed-Point Theorem, g has a fixed-point in this cell, which is a contradiction. Therefore C cannot be g-invariant and we can find a projective plane P_1 which is obtained by a small isotopy of $(P - \operatorname{int} D_1) \cup D$, so that $P_1 \cap gP_1$ has fewer components than $P \cap gP$.

By this procedure we eventually reach a projective plane again denoted by P, with $P \cap gP$ a single curve. The complement of a small g-invariant regular neighbourhood of $P \cup gP$ in RP^3 consists of two 3-cells interchanged by g.

So the action of g is completely characterized and is equivalent to an orthogonal action.

THEOREM 8. *Suppose that M is a 3-manifold as in Proposition 3. If there is a free involution acting on M then the quotient is either a lens space or a manifold with the properties in Proposition 3.*

PROOF. Let $M = N \cup Y$ where N is a small regular neighbourhood of a Klein bottle K embedded in M. Let $g\colon M \to M$ be a free involution. We will show that the quotient has either an embedded Klein bottle or a genus 1 Heegaard splitting and this clearly implies the result.

Assume that gK and K are transverse. By exactly the same procedure as in Proposition 7, since K and gK are incompressible the contractible curves in their intersection can be eliminated. Suppose that a component C of $K \cap gK$ is two-sided in K. If T is a small regular neighborhood of C in M then $T - T \cap K$ has two components. Therefore $gK \cap (T - T \cap K) = (gK \cap T) - C$ has two components, and this shows that C is two-sided in gK.

Suppose next that $K \cap gK$ contains two or more two-sided (noncontractible) curves in K. If C_1, C_2 are loops of this type then clearly $C_1 \cup C_2$ bounds annuli A, A' in K, gK respectively. Without loss of generality assume $K \cap \operatorname{int} A' = \varnothing$. Exactly one of the surfaces $(K - \operatorname{int} A) \cup A'$ and $A \cup A'$ is a Klein bottle, which we denote by K_1. Suppose C_1 is g-invariant and let $\pi\colon M \to M_0$ be the quotient of M by the action of g. By the argument on p. 14 of **[13]** (cf. also p. 44 of **[12]**) this case can only occur if $\pi(C_1)$ is orientation-reversing in M_0, i.e. M_0 is nonorientable. But M_0 is closed with finite fundamental group so this gives a contradiction.

Therefore neither C_1 nor C_2 can be g-invariant. If $C_1 \neq gC_2$ then after separating K_1 slightly from gK_1, we see that $K_1 \cap gK_1$ has less components than $K \cap gK$. On the other hand, if $C_1 = gC_2$ then we can choose notation so that $gA = A'$. In this case if $K_1 = (K - \operatorname{int} A) \cup A'$ then again after a small isotopy, $K_1 \cap gK_1$ has fewer curves than $K \cap gK$. Finally, if $K_1 = A \cup A'$ then K_1 is g-invariant and the result follows, since M_0 contains a Klein bottle.

So we have established that for suitable choice of K, $K \cap gK$ includes at most one two-sided curve. Assume $K \cap gK$ has exactly one such curve C. Then C must be g-invariant, which gives a contradiction. Consequently it suffices to assume $K \cap gK$ contains only one-sided curves.

Case 1. *$K \cap gK$ is a single curve C.*

Let T be a small g-invariant regular neighbourhood of C, so that $K \cap \partial T$ and $gK \cap \partial T$ are single curves, C_1 and gC_1 respectively. Let A be an annulus on ∂T between C_1 and gC_1. Then $K_1 = (K - \operatorname{int} T) \cup A \cup (gK - \operatorname{int} T)$ is an embedded Klein bottle in M. Since M_0 is orientable, g is orientation-preserving on T and on ∂T. Therefore A cannot be g-invariant, because g interchanges the curves of ∂A. Consequently we can separate K_1 slightly from gK_1 so that $K_1 \cap gK_1$ consists of two one-sided curves.

Case 2. $K \cap gK = C \cup gC$ (*where* C *is one-sided*).

Let T be a small regular neighbourhood of C (with $T \cap gT = \emptyset$). Then $\pi(T)$ is a solid torus in M_0 with $\pi(K \cap T)$ equal to a properly embedded Möbius band. Let $K - \operatorname{int} T - \operatorname{int} gT = A$ and denote the closures of the components of $M - \operatorname{int} T - \operatorname{int} gT - K - gK$ by Y_1 and Y_2.

The well-known argument that a properly embedded, incompressible annulus in a solid torus is parallel into the boundary shows that either Y_1 or Y_2 is a solid torus, with a meridian disk D_1 which intersects A and gA each in a single arc. We choose notation so that this is true for Y_1. There are two possibilities:

(1) Y_1 *and* Y_2 *are both g-invariant.*

Let C' be a component of ∂A. Then $\pi(D_1)$ is a meridian disk for the solid torus $\pi(Y_1)$ (because M_0 is orientable) and the curves $\pi(C')$, $\partial\pi(D_1)$ have intersection number ± 2 in $\partial\pi(Y_1)$. So there is a Möbius band B embedded properly in $\pi(Y_1)$ with $\partial B = \pi(C')$. Consequently $B \cup \pi(K \cap T)$ is a nonsingular Klein bottle in M_0.

(2) *g interchanges* Y_1 *and* Y_2.

In this case both Y_1 and Y_2 are solid tori, with meridian disks D_1 and gD_1 which both cross A and gA each at single arcs. Therefore it is easy to see that $Y_1 \cup Y_2$ is homeomorphic to $S^1 \times S^1 \times I$. Consequently by [9], $\pi(Y_1 \cup Y_2)$ is homeomorphic to the twisted line-bundle over a Klein bottle. This proves that M_0 contains a Klein bottle.

Case 3. $K \cap gK = C_1 \cup C_2$, *with both curves g-invariant* (*and one-sided*).

Let T_1 and T_2 be small g-invariant regular neighbourhoods of C_1 and C_2. Define A, Y_1, Y_2 as in Case 2, using T_1 and T_2 instead of T and gT. Exactly as in Case 1, the two annuli on ∂T_1 between the curves $K \cap \partial T_1$ and $gK \cap \partial T_1$ cannot be g-invariant. Therefore it follows that $g\colon Y_1 \to Y_2$ is the only possibility. As in (2) of Case 2 above, we find that $Y_1 \cup Y_2$ is homeomorphic to $S^1 \times S^1 \times I$. Consequently the torus ∂T_1 gives a g-invariant Heegaard splitting of M. This establishes that M_0 has a Heegaard splitting of genus 1 and is a lens space.

COROLLARY 9. *A free action of* Z_8 *or* $Q(2^k)$, $k > 3$, *on* S^3 *is equivalent to an orthogonal action.*

PROOF. Suppose first that $G = Z_8$ or $Q(8)$ and G acts freely on S^3. Then there is a normal subgroup Z_4 of G and by Proposition 7, the quotient of S^3 by Z_4 is $L(4, 1)$. Now this is a manifold of the type in Proposition 3. Let g be the free involution on $L(4, 1)$ induced by the action of G on S^3. Then by

Theorem 8, the quotient of $L(4, 1)$ by g is either a lens space or a manifold with the properties in Proposition 3. But this is clearly the orbit space for the action of G on S^3. Consequently the quotient of S^3 by G is a Seifert manifold, by Proposition 3. Now the action of G is equivalent to an orthogonal action if and only if its orbit space can be Seifert fibered (see **[10]**). Therefore the result is proved.

For $k > 3$ the result follows by induction on k. Suppose $Q(2^k)$ acts freely on S^3. The action of the normal subgroup $Q(2^{k-1})$ is equivalent to an orthogonal action by the inductive assumption. So the quotient of S^3 by the action of $Q(2^{k-1})$, which we will denote by M, is a Seifert manifold. Now it is easy to show that because $\pi_1(M) = Q(2^{k-1})$, M has S^2 as orbit surface and 3 exceptional fibres of multiplicity 2, 2, p, with $p > 1$ (cf. [7] or **[10]**). Therefore M is a manifold of the kind in Proposition 3. There is a free involution on M induced by the action of $Q(2^k)$ on S^3. Then by Theorem 8, the quotient of M by the involution is a Seifert manifold. Since this is just the orbit space for $Q(2^k)$, the proof is complete.

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